Q. If we have a "nice" function $f(x)$, how do we construct a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ which converges to $f(x)$ near $x = a$?

Q. First, why do we care?

Computation: calculators only $+, -, \times, \div$ whole numbers, it can't really do $\sin(x)$

Q. Okay, now answer the question.

(1) Relate (via algebra or calculus) $f(x)$ to $\frac{1}{1-x}$ then use the power series expansion for $\frac{1}{1-x}$

(2) Something else

Note: Here are some instances of (1)

Ex 1 $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ if $x \in (-1, 1)$

Ex 2 $\dfrac{1}{1+x^2} = \displaystyle\sum_{k=0}^{\infty} (-1)^k x^{2k}$ for $x \in (-1, 1)$

Ex 3 $\ln(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$ for $x \in (0, 2)$

Ex 4 $\arctan(x) = \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} x^{2m+1}$ for $x \in (-1, 1)$

Q. What makes it possible find power series for these functions? For which "nice" functions doesn't this approach work?

- Can do the above examples because they relate to some rational function
- Won't work for e^x, $\sin(x)$, $\cos(x)$

Q. What is a better approach for finding the coefficients of a power series representation of $f(x)$? Suppose $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for all x in $(a-R, a+R)$.

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

$$\rightarrow \boxed{f(a)} = c_0 + c_1 \cdot 0 + c_2 \cdot 0 + c_3 \cdot 0 + \cdots$$

$$= \boxed{c_0}$$

$$\rightarrow f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

$$\hookrightarrow \boxed{f'(a)} = c_1 + 2c_2 \cdot 0 + 3c_3 \cdot 0 + \cdots$$

$$= \boxed{c_1}$$

$$\rightarrow f''(x) = 2c_2 + 3 \cdot 2 \cdot c_3(x-a) + 4 \cdot 3 c_4(x-a)^2 + \cdots$$

$$\hookrightarrow f''(a) = 2c_2 + 3 \cdot 2 \cdot c_3 \cdot 0 + 4 \cdot 3 \cdot c_4 \cdot 0 + \cdots$$

$$= 2c_2 \rightarrow \boxed{c_2 = \frac{f''(a)}{2}}$$

$$\rightarrow f'''(x) = 3 \cdot 2 \cdot c_3 + 4 \cdot 3 \cdot 2 \cdot c_4(x-a) + \cdots$$

$$\hookrightarrow f'''(a) = 3 \cdot 2 \cdot c_3 + 4 \cdot 3 \cdot 2 \cdot c_4 \cdot 0 + \cdots$$

$$= 3 \cdot 2 \cdot c_3 \rightarrow c_3 = \frac{f'''(a)}{3 \cdot 2}$$

$$\rightarrow f^{(n)}(x) = n! \cdot c_n + (n+1) \cdots 2 \cdot c_{n+1}(x-a) + \cdots$$

$$\hookrightarrow f^{(n)}(a) = n! \, c_n \rightarrow \boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

Def: Suppose f is a function that is smooth at $x = a$ (i.e. $f(a), f'(a), f''(a), f^{(3)}(a), \ldots$ all exist). The n^{th} Taylor coefficient for f at a is $\frac{f^{(n)}(a)}{n!}$ and the Taylor series for f at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

If $a = 0$ it is sometimes called a Maclaurin series.

"Thm:" If $f(x)$ is "nice," then the Taylor series for $f(x)$ converges to $f(x)$ on its radius of convergence.

"Proof:" To come later with specific examples for f(x)

put somewhere more useful

Def: The kth partial sum of the Taylor Series for f centered at a, namely

$$P_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}\cdot(x-a)^k$$

is called the k^{th} Taylor polynomial

Ex 5 Find the Taylor series for $f(x) = e^x$ centered at 0, and find its interval of convergence.

Want: c_n for all n

$$c_n = \frac{f^{(n)}(0)}{n!}$$

Note: $f^{(n)}(x) = e^x \rightarrow f^{(n)}(0) = 1$

$$\rightarrow c_n = \frac{1}{n!}$$

$$\Rightarrow \text{Taylor series for } e^x \text{ is } \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$I = (-\infty, \infty)$$

Ex 6 Find the Taylor series for $f(x) = \cos(x)$ centered at 0 and find its interval of convergence.

$f(x) = \cos(x)$; $f'(x) = -\sin(x)$; $f''(x) = -\cos(x)$; $f'''(x) = \sin(x)$

$$\rightarrow f^{(n)}(x) = \begin{cases} \cos(x) & n = 4k \\ -\sin(x) & n = 4k+1 \\ -\cos(x) & n = 4k+2 \\ \sin(x) & n = 4k+3 \end{cases}$$

$$\rightarrow f^{(n)}(0) = \begin{cases} 1 & n = 4k \\ 0 & n = 4k+1 \\ -1 & n = 4k+2 \\ 0 & n = 4k+3 \end{cases}$$

With $c_n = \dfrac{f^{(n)}(0)}{n!}$, the Taylor series is

$$\sum_{n=0}^{\infty} c_n \cdot x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + C_6 x^6 + C_7 x^7 + \cdots$$

$$= \frac{1}{0!} + \frac{0}{1!} \cdot x + \frac{-1}{2!} x^2 + \frac{0}{3!} x^3 + \frac{1}{4!} x^4 + \frac{0}{5!} x^5 + \frac{-1}{6!} x^6 + \frac{0}{7!} x^7 + \cdots$$

$$= 1 + \frac{-1}{2!} x^2 + \frac{1}{4!} x^4 + \frac{-1}{6!} x^6 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$I = (-\infty, \infty)$$

Ex 7 Find the Taylor series for $f(x) = \sin(x)$ centered at 0 and find its interval of convergence.

↓ a good definition of e^x

Cool Stuff:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\to \frac{d}{dx}\left(e^x\right) = \frac{d}{dx}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)$$

$$= 0 + 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$= e^x$$

Recall: $i = \sqrt{-1}$

$\hookrightarrow i^0 = 1,\ i^1 = i,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1$

$$\Rightarrow i^n = \begin{cases} 1 & n = 4k \\ i & n = 4k+1 \\ -1 & n = 4k+2 \\ -i & n = 4k+3 \end{cases}$$

Consider e^{it} for $t \in \mathbb{R}$:

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \frac{i^0 t^0}{0!} + \frac{it}{1!} + \frac{i^2 t^2}{2!} + \frac{i^3 t^3}{3!} + \frac{i^4 t^4}{4!} + \cdots$$

$$= 1 + \frac{it}{1!} + \frac{(-1)t^2}{2!} + \frac{(-i)t^3}{3!} + \frac{t^4}{4!} + \frac{it^5}{5!} + \frac{(-1)t^6}{6!} + \frac{(-i)t^7}{7!} + \cdots$$

$$= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \cdots$$

$$+ \frac{it}{1!} - \frac{it^3}{3!} + \frac{it^5}{5!} - \frac{it^7}{7!} + \frac{it^9}{9!} - \cdots$$

$$= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \cdots$$

$$+ i\left(\frac{t}{1!} - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \cdots\right)$$

$$= \cos(t) + i\sin(t)$$

$$\Rightarrow e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 \quad (e^{i\pi} + 1 = 0)$$

$$\Rightarrow e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$$

$$\Rightarrow e^{\frac{i\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$$

Q. So why is it that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$?

Want to show: $e^x = \lim_{k\to\infty} \sum_{n=0}^{k} \frac{x^n}{n!}$

$$e^x - \lim_{k\to\infty} \sum_{n=0}^{k} \frac{x^n}{n!} = 0$$

$$\lim_{k\to\infty} \left(e^x - \sum_{n=0}^{k} \frac{x^n}{n!} \right) = 0$$

$$\underbrace{e^x - \sum_{n=0}^{k} \frac{x^n}{n!}}_{R_k(x)}$$

Thm: (Taylor's Theorem with Remainder) Suppose f is ~~$(n+1)$~~ $k+1$-times differentiable on an interval $(a-R, a+R)$ ~~r~~.

Given $x \in (a-R, a+R)$, define the kth remainder by

$$R_k(x) := f(x) - p_k(x)$$

where $p_k(x)$ is the kth Taylor polynomial for $f(x)$ at a.

(a) There is a number $c \in [x, a]$ (or $[a, x]$ if $x \geqslant a$) such that $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

(b) If $|f^{(n+1)}(z)| \leqslant M$ for every $z \in [x, a]$ (or $[a, x]$ if $x \geqslant a$), then

$$|R_k(x)| \leq \frac{M}{(k+1)!} |x-a|^{k+1}$$

Ex 8 Use the Maclaurin series for $f(x) = e^x$ to estimate (i) e^1, (ii) e^5, (iii) e^{15} each with error < 0.0001.

$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

(i) $e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}$

↳ $a = 0$, $x = 1$

Want: $|R_k(1)| \leq 0.0001$

To use Taylor's estimate, need to bound $f^{(k)}(z)$ on $[a, x] = [0, 1]$

$f^{(k)}(z) = e^z$ is bounded above by e for $z \in [0,1]$.

Know: $|R_k(1)| \leq \frac{e}{(k+1)!}(1-0)^{k+1} = \frac{e}{(k+1)!}$

If $\frac{e}{(k+1)!} < 0.0001$, so is $|R_k(1)|$.

$\frac{e}{(k+1)!} < 0.0001 \leftrightarrow \frac{e}{0.0001} < (k+1)! \leftrightarrow 10000e < (k+1)!$

Trial and error: $k = 7$

$P_7(1) = \sum_{n=0}^{7} \frac{1}{n!} \approx$

(ii) $a = 0, \quad x = 5$

Want $|R_k(5)| < 0.0001$

Goal: bound $f^{(k+1)}(z)$ on $[0, 5]$

e^z

$|e^z| \leq e^5$ on $[0, 5]$

Know: $|R_k(5)| \leq \frac{e^5}{(k+1)!}(5-0)^k = \frac{e^5 \cdot 5^k}{(k+1)!}$

Find k s.t. $\frac{e^5 \cdot 5^k}{(k+1)!} < 0.0001 \iff 10000e^5 < \frac{(k+1)!}{5^k}$

Trial and error: $k = 21$

$e^5 \approx p_{21}(5) = \sum_{n=0}^{21} \frac{5^n}{n!} \approx$

(iii) Want: $|R_k(15)| < 0.0001$

Note that $|f^{(k+1)}(z)| \leq e^{15}$ on $[0, 15]$

Know: $|R_k(15)| \leq \frac{e^{15}}{(k+1)!} \cdot 15^k$

Goal: Find k s.t. $\frac{e^{15}}{(k+1)!} \cdot 15^k < 0.0001$

Trial-and-error: $k = 56$

$e^{15} \approx p_{56}(15) = \sum_{n=0}^{56} \frac{15^n}{n!} \approx$

Summary: (a) If f is smooth at a, then we can compute the Taylor series at a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n$$

(b) Using the ratio test, we can find a value R such that the Taylor series converges for all x in $(a-R, a+R)$.

(c) Given x in $(a-R, a+R)$, Taylor's Theorem tells us:

If $|f^{(k+1)}(z)| \le M$ for each $z \in [a;x]$ or $[x,a]$, then $|R_k(x)| \le \frac{M}{(k+1)!}\cdot|x-a|^k$

Observe: The greater the distance between x and a, the larger M might need to be and the larger $|x-a|$ is, so k needs to be extra large in order to approximate well.

Ex 9 Why is the Maclaurin series for $\cos(x)$ equal to $\cos(x)$?

$f(x) = \cos(x)$

Fix any $x \in \mathbb{R}$

Goal: $\lim_{k\to\infty} |R_k(x)| = 0$

Note: $0 \le |R_k(x)|$, so $0 \le \lim_{k\to\infty} |R_k(x)|$

Note: $f^{(k)}(x) = \pm \sin(x)$ or $\pm \cos(x)$, depending on k.

But, no matter x or k, $|f^{(k)}(x)| \leq 1$

By Taylor's estimate, $|R_k(x)| \leq \frac{1}{(k+1)!} \cdot |x|^{k+1}$

But $\lim_{k \to \infty} \frac{|x|^{k+1}}{(k+1)!} = 0$, so $\lim_{k \to \infty} |R_k(x)| \leq 0$

$\therefore \quad \lim_{k \to \infty} |R_k(x)| = 0$

$\therefore \quad \lim_{k \to \infty} |\cos(x) - p_k(x)| = 0$

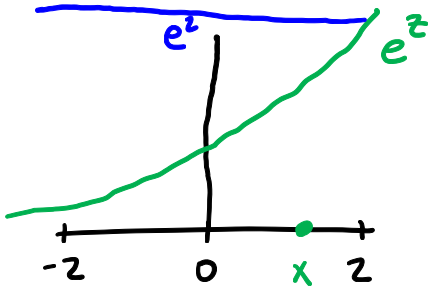
$\therefore \quad \cos(x) = \lim_{k \to \infty} p_k(x)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

- Find k s.t. M. series of e^x is within 0.001 of e^x on $[-2,2]$

Know: If $|f^{(k+1)}(z)| \le M$ for all $z \in I$, then $|R_k(x)| \le \frac{M}{(n+1)!} \cdot |x-0|^{n+1}$

e^z



Want to bound e^z on I

$e^z \le \underset{M}{e^2}$ on I $\rightarrow$ $|R_k(x)| \le \frac{e^2}{(k+1)!}|x|^{k+1} \overset{?}{<} 0.001$

$\le \frac{e^2 2^{k+1}}{(k+1)!}$

$\frac{e^2 2^{k+1}}{(k+1)!} < 0.001$ when $k = 10$

$\Rightarrow |R_{10}(x)| < 0.001$ for all $x \in [-2,2]$